Off-forward parton distributions and Shuvaev's transformations

Jens D. Noritzsch*
Institut für Physik, Universität Dortmund, D-44221 Dortmund, Germany
(Received 4 April 2000; published 1 August 2000)

We review Shuvaev's transformations that relate off-forward parton distributions (OFPDs) to the so-called effective forward parton distributions (EFPDs). The latter evolve like conventional forward partons. We express nonforward amplitudes, depending on OFPDs, directly in terms of EFPDs and construct a model for the EFPDs, which allows us to consistently express them in terms of the conventional forward parton distributions and nucleon form factors. Our model is self-consistent for arbitrary x, ξ, μ, and t.

PACS number(s): 13.60.Hb, 11.15.Bt, 13.60.Le, 14.20.Dh

I. INTRODUCTION

The treatment of nonforward high-energy processes, such as deeply virtual Compton scattering (DVCS) and hard exclusive electroproduction of vector mesons, in perturbative QCD gives rise to a new class of parton distributions, the so-called skewed parton distributions (SPDs) [1–8] and double distributions (DDs) [9,8], generalizing the well-known conventional parton distributions [10–12] and, at the same time, the nucleon form factors. In the following we restrict ourselves to the off-forward parton distributions (OFPDs) introduced by Ji [3–5], which are equivalent to nonforward [7,8] or off-diagonal [6] parton distributions. Therefore, our results can be easily generalized to nonforward and off-diagonal parton distributions.

The off-forward parton distributions $H_p(x,\xi,t,\mu)$, which parametrize nonforward matrix elements of light cone bilocal operators $\langle P'|\mathcal{O}_p(-n/2,n/2)|P\rangle|_{n^2=0}$, depend on the momentum fraction x of the average nucleon momentum $\overline{P} := (P+P')/2$, which the initial state parton p carries, on the "skewedness" $\xi = -n\cdot\Delta/2n\cdot\overline{P}$ with $\Delta := P'-P$, on the momentum transfer invariant $t := \Delta^2$, and on the renormalization scale μ. For vanishing ξ and t, they are identical to the usual forward parton distributions. Detailed reviews on off-forward parton distributions can be found, e.g., in Refs. [8,5].

Recently, Shuvaev [13] demonstrated that the off-forward parton distributions can be related (at least in leading order) by simple transformations to so-called effective forward parton distributions (EFPDs), the renormalization scale dependence of which is governed by the conventional forward evolution equations. These relations have led to some progress in determining the shape of the off-forward parton distributions for small values of ξ [14], since the EFPDs can be identified with the usual partons for small values of ξ and arbitrary scale μ.

In the present paper, we express nonforward amplitudes directly in terms of effective forward parton distributions. Furthermore, we define a family of self-consistent models for EFPDs, in which the effective forward parton distributions are obtained from the conventional forward parton distributions and nucleon form factors at arbitrary scale μ.

In the next section we briefly review the basic properties of the off-forward parton distributions, and we define the effective forward parton distributions. In Sec. III, we recalculate Shuvaev's inverse transformations, which relate the EFPDs to the OFPDs, and we derive their support in x, which is not identical to $-1\leqslant x\leqslant 1$ as for the conventional forward parton distributions.[1] In Sec. IV, Shuvaev's transformation is brought into a form that is convenient for a further analytical and numerical treatment. In Sec. V, we connect the effective forward parton distributions directly to nonforward amplitudes, and we briefly discuss the reliability of simple approximative formulas. In Sec. VI, we introduce our model. Finally, in Sec. VII, we summarize our results, and we draw the conclusions.

II. OFF-FORWARD AND EFFECTIVE PARTON DISTRIBUTIONS

The long-distance behavior of hard scattering processes, which is not calculable in (QCD) perturbation theory, is factorized in matrix elements of light-cone bilocal operators. A Fourier transformation of diagonal matrix elements results in the conventional quark and gluon densities $q(x)$ and $g(x)$. Analogously, the off-forward parton distributions are defined by nonforward matrix elements:

$$\left\langle P',S'\left|\overline{\psi}_q\left(-\frac{n}{2}\right)\not{n}\mathcal{G}\psi_q\left(\frac{n}{2}\right)\right|P,S\right\rangle\Big|_{n^2=0} = \overline{U}(P',S')\not{n}U(P,S)\int_{-1}^{+1}e^{-ix(n\cdot\overline{P})} \times H_q(x,\xi,t)\,dx+\mathcal{O}(\Delta), \tag{1a}$$

$$\left\langle P',S'\left|F^a_{\mu\lambda}\left(-\frac{n}{2}\right)n^\mu n^\nu\mathcal{G}_{ab}F^{b\lambda}{}_\nu\left(\frac{n}{2}\right)\right|P,S\right\rangle\Big|_{n^2=0} = \frac{1}{2}\overline{U}(P',S')\not{n}U(P,S)(n\cdot\overline{P})\int_{-1}^{+1}e^{-ix(n\cdot\overline{P})} \times H_g(x,\xi,t)\,dx+\mathcal{O}(\Delta), \tag{1b}$$

[1] We use throughout parton distributions with both signs of x, i.e., $q(-x) = -\overline{q}(x)$ and $g(-x) = -g(x)$.

*Email address: jensn@hal1.physik.uni-dortmund.de

where $\mathcal{G}_{(ab)}$ is the Wilson gauge link. Note that we have an additional factor of x in the definition of the gluon distribution compared to the original definition of Ji [3–5], $H_g = xH_g^{\rm Ji}$, which removes an ''artificial'' singularity for finite ξ [8]. Due to time reversal invariance and hermiticity, Ji's OFPDs are even functions of ξ, so it is sufficient to treat only positive values of ξ. The different $\mathcal{O}(\Delta)$ contributions can be found, for example, in Refs. [8,5,15]. For vanishing Δ the off-forward parton distributions reduce to the diagonal partons:

$$H_q(x,0,0) = q(x), \tag{2a}$$

$$H_g(x,0,0) = xg(x). \tag{2b}$$

The renormalization of the defining operators leads to a scale dependence of the off-forward parton distributions. The evolution of the OFPDs takes a simple form at the one-loop level for the Gegenbauer moments,

$$G_n^q(\xi,t,\mu) := \frac{2^n[n!]^2}{(2n+1)!}\int_{-1}^{+1}\xi^n C_n^{(3/2)}\left(\frac{x}{\xi}\right)H_q(x,\xi,t,\mu)\,dx, \tag{3a}$$

$$G_n^g(\xi,t,\mu) := \frac{3\cdot 2^n(n-1)!n!}{(2n+1)!}\int_{-1}^{+1}\xi^{n-1}C_{n-1}^{(5/2)}\left(\frac{x}{\xi}\right) \times H_g(x,t,\mu,\xi)\,dx, \tag{3b}$$

since they evolve exactly as the Mellin moments in the diagonal case [8]. For example, for the Gegenbauer moments of the nonsinglet off-forward quark distributions one has

$$G_n^{q,\rm ns}(\xi,t,\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_{0n}/2\beta_0} G_n^{q,\rm ns}(\xi,t,\mu_0), \tag{4}$$

where γ_{0n} and β_0 are the leading coefficients of the nonsinglet anomalous dimension and the beta function. This fact allows the definition of effective forward parton distributions, whose Mellin moments equal the corresponding Gegenbauer moments of the OFPDs [13]:

$$\int_{-1}^{+1} x^n q_{\xi,t}(x,\mu)\,dx = G_n^q(\xi,t,\mu), \tag{5a}$$

$$\int_{-1}^{+1} x^n g_{\xi,t}(x,\mu)\,dx = G_n^g(\xi,t,\mu). \tag{5b}$$

Their scale dependence is governed by the conventional evolution equations, and they reduce to the diagonal quark and gluon densities for $\xi,t \to 0$. The crucial point is that the effective and off-forward parton distributions can be related to each other by Shuvaev's transformations [13]. As these transformations do not depend on the momentum transfer invariant t and the scale μ, we can safely skip them in the following.

III. SHUVAEV'S INVERSE TRANSFORMATION

We start with connecting the effective forward parton distribution to the off-forward ones. This can be done by Shuvaev's inverse integral transformation [13]:

$$q_\xi(x) = \int_{-1}^{+1}\mathcal{K}_q^{-1}(x,\xi;y)H_q(y,\xi)\,dy, \tag{6a}$$

$$g_\xi(x) = \int_{-1}^{+1}\mathcal{K}_g^{-1}(x,\xi;y)H_g(y,\xi)\,dy. \tag{6b}$$

We briefly sketch the main steps of the derivation of the integral kernels $\mathcal{K}_{q,g}^{-1}(x,\xi;y)$ in Ref. [13], in order to determine the support properties of the EFPDs. The calculation is based on the formal inversion of the Mellin moments in Eq. (5):

$$q_\xi(x) = -\frac{1}{\pi}\,\mathrm{disc}\sum_{n=0}^{\infty}\frac{G_n^q(\xi)}{x^{n+1}}, \tag{7a}$$

$$g_\xi(x) = -\frac{1}{\pi}\,\mathrm{disc}\sum_{n=1}^{\infty}\frac{G_n^g(\xi)}{x^{n+1}}, \tag{7b}$$

with

$$\mathrm{disc}\,F(x) = \frac{1}{2i}\lim_{\varepsilon\to 0}[F(x+i\varepsilon)-F(x-i\varepsilon)].$$

The Gegenbauer moments $G_n^{q,g}(x)$ are defined in Eq. (3). The factorial functions in Eq. (3) are replaced using the integral representation of the beta function {Eqs. (6.1.18) and (6.2.1) in Ref. [16]}. We obtain

$$q_\xi(x) = \int_{-1}^{+1}\left[-\frac{1}{\pi}\,\mathrm{disc}\int_{-1}^{+1}\frac{1}{2x\sqrt{1-s}}\sum_{n=0}^{\infty}C_n^{(3/2)}\left(\frac{y}{\xi}\right)\times\left(\frac{s\xi}{2x}\right)^n ds\right]H_q(y,\xi)\,dy, \tag{8a}$$

$$g_\xi(x) = \int_{-1}^{+1}\left[-\frac{1}{\pi}\,\mathrm{disc}\int_{-1}^{+1}\frac{3\sqrt{1-s}}{2x^2}\sum_{n=0}^{\infty}C_n^{(5/2)}\left(\frac{y}{\xi}\right)\times\left(\frac{s\xi}{2x}\right)^n ds\right]H_g(y,\xi)\,dy. \tag{8b}$$

The expressions in square brackets are the integral kernels $\mathcal{K}_{q,g}^{-1}(x,\xi;y)$. Before we state their final form, we have to look at the generating functions of the Gegenbauer polynomials {Eq. (22.9.3) in Ref. [16]}:

$$\exp(-\nu\,\mathrm{Log}(1-2wz+z^2)) = \sum_{n=0}^{\infty}C_n^{(\nu)}(w)z^n. \tag{9}$$

The generating functions on the left-hand side analytically continue the power series on the right-hand side to the complete complex plane. In Fig. 1 we show the circles of con-

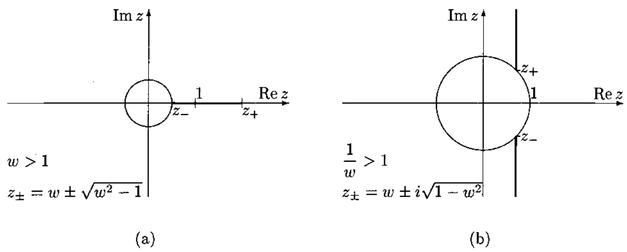

FIG. 1. Circles of convergence of the power series in Eq. (9) for positive *w*. The corresponding figures for negative *w* are received by mirroring at the vertical axis. The thick lines show the discontinuities of the generating functions.

vergence of the power series and the discontinuities of the generating functions, which arise from negative arguments of the complex logarithm. We see that we have to distinguish two cases: $|y|>\xi$, which corresponds to the parton-distribution-like region of the OFPDs, and $|y|<\xi$, which corresponds to the meson-wave-function-like region. Let us begin with the latter case. One might think that for $|y|<\xi$ one has no contribution to the integral kernels $\mathcal{K}_{q,g}^{-1}(x,\xi;y)$ because the generating functions are analytical for any finite and real z. However, the discontinuity at $z=\infty$ produces delta functions and their derivatives $\delta^{(n)}(x)$. From the circle of convergence we find that their contribution is in any case restricted to $|x|<\xi/2$ in the effective forward parton distributions. For $|y|>\xi$ the discontinuity can be easily calculated. We face strong singularities in the generating functions at the end points of the cut, therefore, we have to take derivatives of less singular functions, so that the s-integral in Eq. (8) is convergent. An examination of the circle of convergence shows that

$$q_{\xi}(x)=0 \quad \text{and} \quad g_{\xi}(x)=0, \quad \text{if } |x|>x_b:=\tfrac{1}{2}(1+\sqrt{1-\xi^2}). \tag{10}$$

This defines the support area of the EFPDs, which is shown in Fig. 2. Finally, the complete result for the integral kernels of Shuvaev's inverse transformation is

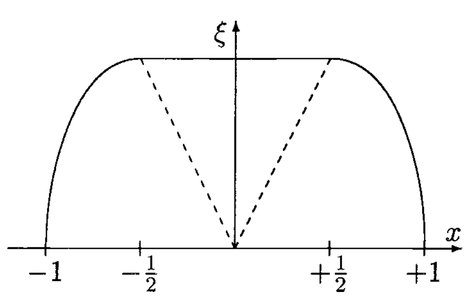

FIG. 2. Support area of the effective forward parton distributions. The dashed lines correspond to $x=\pm\xi/2$. The contribution of the meson-wave-function-like part of the OFPDs is restricted to the region between the dashed lines.

$$\begin{aligned}\mathcal{K}_q^{-1}(x,\xi;y) &= \frac{2x}{\pi|\xi|}\frac{\partial}{\partial y}\int_0^1 \frac{\theta(-s^2+4sxy/\xi^2-4x^2/\xi^2)}{s\sqrt{(1-s)(-s^2+4sxy/\xi^2-4x^2/\xi^2)}}ds \\ &\quad +\theta\left(1-\left|\frac{y}{\xi}\right|\right)\sum_{n=0}^{\infty}(-1)^n\frac{2^n[n!]^2}{(2n+1)!} \\ &\quad \times \xi^n C_n^{(3/2)}\left(\frac{y}{\xi}\right)\delta^{(n)}(x),\end{aligned} \tag{11a}$$

$$\begin{aligned}\mathcal{K}_g^{-1}(x,\xi;y) &= \frac{4x}{\pi|\xi|}\frac{\partial^2}{\partial y^2}\int_0^1 \frac{\theta(-s^2+4sxy/\xi^2-4x^2/\xi^2)\sqrt{1-s}}{s^2\sqrt{-s^2+4sxy/\xi^2-4x^2/\xi^2}}ds \\ &\quad +\theta\left(1-\left|\frac{y}{\xi}\right|\right)\sum_{n=1}^{\infty}(-1)^n\frac{3\cdot 2^n(n-1)!n!}{(2n+1)!} \\ &\quad \times \xi^{n-1} C_{n-1}^{(5/2)}\left(\frac{y}{\xi}\right)\delta^{(n)}(x).\end{aligned} \tag{11b}$$

We cannot carry out the derivatives, which would lead to infinite contributions from the derivatives of the theta functions and divergent integrals. The infinite sums represent the contribution of the meson-wave-function-like part of the off-forward parton distributions and do not appear in Ref. [13], since the discontinuity at $z=\infty$, resp. $x=0$, was overlooked. The occurrence of delta functions and their derivatives in the effective forward parton distributions are comparable to meson-exchange-type contributions in double distributions [17,18].

The expressions for the integral kernels $\mathcal{K}_{q,g}^{-1}(x,\xi;y)$ show that the Gegenbauer moment inversion is not practicable, in general. Therefore, it can generally not be used for a simple solution of the evolution equations of off-forward parton distributions.

IV. SHUVAEV'S TRANSFORMATION

As we will later see, the predictive power of the formalism lies in relating the effective forward parton distributions to the off-forward ones by Shuvaev's integral transformation

$$H_q(x,\xi)=\int_{-1}^{+1}\mathcal{K}_q(x,\xi;y)q_{\xi}(y)\,dy, \tag{12a}$$

$$H_g(x,\xi)=\int_{-1}^{+1}\mathcal{K}_g(x,\xi;y)g_{\xi}(y)\,dy. \tag{12b}$$

The full derivation of the integral kernels $\mathcal{K}_{q,g}(x,\xi;y)$ can be found in Ref. [13]. We merely state the finite result:

$$\begin{aligned}\mathcal{K}_q(x,\xi;y) &= \frac{1}{\pi\sqrt{|y|}}\frac{\partial}{\partial y}\frac{y}{\sqrt{|y|}}\int_{-1}^{+1}\theta\left(\frac{y(1-s^2)}{x-\xi s}-1\right) \\ &\quad \times\sqrt{\frac{x-\xi s}{y(1-s^2)-x+\xi s}}\,ds,\end{aligned} \tag{13a}$$

$$\mathcal{K}_g(x,\xi;y)=\frac{1}{\pi\sqrt{|y|}}\frac{\partial}{\partial y}\sqrt{|y|}\int_{-1}^{+1}\theta\left(\frac{y(1-s^2)}{x-\xi s}-1\right)\times\sqrt{\frac{(x-\xi s)^3}{y(1-s^2)-x+\xi s}}\,ds. \tag{13b}$$

Again, performing the derivatives would give divergent integrals and infinite contributions from the end points. The derivatives of the theta function give rise to the ''suspicious overall sign'' that is mentioned in Ref. [18]. Equation (13) is equivalent to previous results presented in Refs. [14,18].

It is useful to express the integral kernels in terms of standard elliptic integrals, because it is then possible to perform the derivatives analytically. First, we give the symmetry properties of the integral kernels:

$$\mathcal{K}_q(x,-\xi;y)=\mathcal{K}_q(x,\xi;y),\quad \mathcal{K}_q(x,\xi;-y)=+\mathcal{K}_q(-x,\xi;y), \tag{14a}$$

$$\mathcal{K}_g(x,-\xi;y)=\mathcal{K}_g(x,\xi;y),\quad \mathcal{K}_g(x,\xi;-y)=-\mathcal{K}_g(-x,\xi;y). \tag{14b}$$

Of course, $\mathcal{K}_{q,g}(x,\xi;y)$ obey the fundamental $(-\xi\leftrightarrow\xi)$-symmetry of Ji's off-forward parton distributions. The relations on the right-hand side show that it is sufficient to restrict the calculation to positive values of y. We define

$$a:=\frac{x}{\xi}-\frac{\xi}{2y}-\frac{1}{2y}\sqrt{4y^2-4yx+\xi^2},$$

$$b:=\frac{x}{\xi}-\frac{\xi}{2y}+\frac{1}{2y}\sqrt{4y^2-4yx+\xi^2}, \tag{15}$$

which correspond up to the x/ξ, which we added for convenience, to the zeroes of the denominator in the square root in Eq. (13). As the radicand has to be positive, we must further restrict the possible range of y to

$$y>x_a:=\tfrac{1}{2}(x+\sqrt{x^2-\xi^2}),\quad \text{if } x\geqslant\xi. \tag{16}$$

With help of the integral tables in Ref. [19] we obtain our final result:

$$\mathcal{K}_q(x,\xi;y)=\begin{cases}\delta(y-x_a)\sqrt{\frac{1}{x_a}\sqrt{x^2-\xi^2}}\\ \quad+\theta(y-x_a)\frac{\xi}{\pi y^2}\sqrt{\frac{\xi}{yb}}\frac{b}{b-a}\left(R_F\left(0,\frac{a}{b},1\right)-\frac{1}{3}\frac{b+a}{b}R_D\left(0,\frac{a}{b},1\right)\right) & \text{for } x\geqslant\xi,\\ \theta(x+\xi)\frac{\xi}{\pi\xi^2}\sqrt{\frac{\xi}{y(b-a)}}\frac{b}{b-a}\left(R_F\left(0,\frac{-a}{b-a},1\right)-\frac{1}{3}\frac{b+a}{b-a}R_D\left(0,\frac{-a}{b-a},1\right)\right) & \text{for } x<\xi,\end{cases} \tag{17a}$$

$$\mathcal{K}_g(x,\xi;y)=\begin{cases}\delta(y-x_a)\sqrt{\frac{1}{x_a}(x^2-\xi^2)^{3/2}}+\theta(y-x_a)\frac{\xi^2}{\pi y^2}\sqrt{\frac{\xi b}{y}}\frac{b}{b-a}\\ \quad\times\left(\frac{2b-a}{b}R_F\left(0,\frac{a}{b},1\right)-\frac{2}{3}\frac{b^2-ab+a^2}{b^2}R_D\left(0,\frac{a}{b},1\right)\right) & \text{for } x\geqslant\xi,\\ \theta(x+\xi)\frac{\xi^2}{\pi y^2}\sqrt{\frac{\xi(b-a)}{y}}\frac{b}{b-a}\\ \quad\times\left(\frac{2b-a}{b-a}R_F\left(0,\frac{-a}{b-a},1\right)-\frac{2}{3}\frac{b^2-ab+a^2}{(b-a)^2}R_D\left(0,\frac{-a}{b-a},1\right)\right) & \text{for } x<\xi,\end{cases} \tag{17b}$$

where R_F and R_D are Carlson's elliptic integrals of the first and second kind (see, e.g., [20]) with

$$R_F(x,y,z):=\frac{1}{2}\int_0^\infty\frac{1}{\sqrt{(t+x)(t+y)(t+z)}}\,dt, \tag{18}$$

$$R_D(x,y,z):=\frac{3}{2}\int_0^\infty\frac{1}{(t+z)\sqrt{(t+x)(t+y)(t+z)}}\,dt. \tag{19}$$

These integral kernels accumulate the main properties of the x and ξ dependence of off-forward parton distributions. For $x\gg\xi$, the OFPDs essentially look like forward quarks and gluons:

$$\mathcal{K}_q(x,\xi;y)=\delta(y-x)+\frac{1}{x}\mathcal{O}\left(\frac{\xi^2}{x^2}\right), \tag{20a}$$

$$\mathcal{K}_g(x,\xi;y)=x\,\delta(y-x)+\mathcal{O}\left(\frac{\xi^2}{x^2}\right). \tag{20b}$$

The forward evolution concentrates the effective forward parton distributions at $y\sim 0$. Therefore, the small-y behavior of the integral kernels $\mathcal{K}_{q,g}(x,\xi;y)$ reproduces the well-known asymptotic forms of the off-forward valence and singlet quark, and gluon distributions [8]:

$$H_q^v(x,\xi)=\theta\left(1-\left|\frac{x}{\xi}\right|\right)\frac{\xi^2-x^2}{\xi^3}\int_0^\varepsilon\left(\frac{3}{2}+\mathcal{O}\left(\frac{y}{\xi}\right)\right)q_\xi^v(y)\,dy, \tag{21a}$$

$$H_q^s(x,\xi)=\theta\left(1-\left|\frac{x}{\xi}\right|\right)\frac{x(\xi^2-x^2)}{\xi^5}\int_0^\varepsilon\left(\frac{15}{2}+\mathcal{O}\left(\frac{y}{\xi}\right)\right)y q_\xi^s(y)\,dy,$$

$$H_g(x,\xi)=\theta\left(1-\left|\frac{x}{\xi}\right|\right)\frac{(\xi^2-x^2)^2}{\xi^5}\int_0^\varepsilon\left(\frac{15}{8}+\mathcal{O}\left(\frac{y}{\xi}\right)\right)y g_\xi(y)\,dy. \tag{21b}$$

Additionally, these equations prove that the integrals in Eq. (12) are well defined and convergent.

Even the physical interpretation of the different regions in x holds in the formalism of EFPDs. Inserting Eq. (14) in Eq. (12) yields

$$H_q(x,\xi)=\int_0^1(\mathcal{K}_q(x,\xi;y)q_\xi(y)+\mathcal{K}_q(-x,\xi;y)q_\xi(-y))\,dy, \tag{22a}$$

$$H_g(x,\xi)=\int_0^1(\mathcal{K}_g(x,\xi;y)+\mathcal{K}_g(-x,\xi;y))g_\xi(y)\,dy. \tag{22b}$$

For $|x|>\xi$ the off-forward parton distributions are related to corresponding effective forward (anti)partons with a minimum momentum $|x_a|$. For $|x|<\xi$ the picture of a meson wave function is supported by a simultaneous contribution of effective forward partons and antipartons with any momentum y. The different expressions for the integral kernels $\mathcal{K}_{q,g}(x,\xi;y)$ for $|x|\gtrless\xi$ show, analogous to [17], that the off-forward parton distributions are not analytic at $|x|=\xi$. The analyticity of the OFPDs for $x\neq\xi$ requires that the effective forward parton distributions need to be analytic for $|x|\geqslant\xi/2$ only.

V. NONFORWARD AMPLITUDES AND EFPDs

The use of off-forward parton distributions is required in deeply virtual Compton scattering (DVCS) and hard exclusive electroproduction processes. Detailed information can be found in Ref. [5], and references therein. Here, we are only interested in the part of the amplitudes that refers to the OFPDs:

$$\mathcal{A}_q(\xi):=\int_{-1}^{+1}\left(\frac{1}{x-\xi+i\varepsilon}+\frac{1}{x+\xi-i\varepsilon}\right)H_q(x,\xi)\,dx, \tag{23a}$$

$$\mathcal{A}_g(\xi):=\int_{-1}^{+1}\left(\frac{1}{x-\xi+i\varepsilon}+\frac{1}{x+\xi-i\varepsilon}\right)\frac{1}{x}H_g(x,\xi)\,dx, \tag{23b}$$

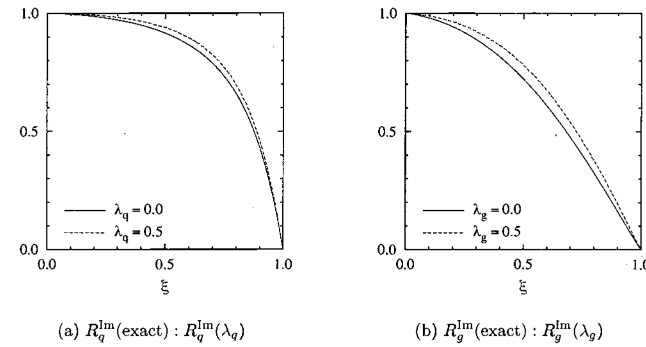

FIG. 3. Comparison of the exact imaginary part to the approximative ratio $R_{q,g}^{\rm Im}(\lambda_{q,g})$ for $q_\xi^s(x),g_\xi(x)\sim x^{-\lambda_{q,g}-1}$.

where we have neglected the $\mathcal{O}(\Delta)$ contributions, analogous to Ref. [14]. The imaginary part of the amplitudes is related to the diagonal elements $H_{q,g}(\xi,\xi)$, which can be expressed by the effective forward parton distributions:

$$-\pi H_q^s(\xi,\xi)={\rm Im}\,\mathcal{A}_q(\xi)=-\int_0^{\sqrt{1-\xi/2x_b}}4\sqrt{1-z^2}\,q_\xi^s\left(\frac{\xi}{2(1-z^2)}\right)dz, \tag{24a}$$

$$-\frac{2\pi}{\xi}H_g(\xi,\xi)={\rm Im}\,\mathcal{A}_g(\xi)=-\int_0^{\sqrt{1-\xi/2x_b}}32z^2\sqrt{1-z^2}\,g_\xi\left(\frac{\xi}{2(1-z^2)}\right)dz, \tag{24b}$$

with the quark singlet $q_\xi^s(x)=q_\xi(x)-q_\xi(-x)$. The imaginary part is essentially dominated by the behavior of the EFPDs around $x\sim\xi/2$. For small values of ξ this region can be accurately described by

$$x q_\xi^s(x)\sim x^{-\lambda_q}, \tag{25a}$$

$$x g_\xi(x)\sim x^{-\lambda_g}. \tag{25b}$$

If we insert Eq. (25) into Eq. (24) and set the upper integration limits to one, we achieve approximation formulas for the imaginary part of the amplitudes in Eq. (23):

$$R_q^{\rm Im}:=\frac{{\rm Im}\,\mathcal{A}_q(\xi)}{q_\xi^s(\xi/2)}\simeq-\frac{2\sqrt{\pi}\,\Gamma(\lambda_q+\frac{5}{2})}{\Gamma(\lambda_q+3)}, \tag{26a}$$

$$R_g^{\rm Im}:=\frac{{\rm Im}\,\mathcal{A}_g(\xi)}{g_\xi(\xi/2)}\simeq-\frac{8\sqrt{\pi}\,\Gamma(\lambda_g+\frac{5}{2})}{\Gamma(\lambda_g+4)}. \tag{26b}$$

A similar ratio was already presented in Ref. [14], where, however, the imaginary part was compared to diagonal partons at $x=2\xi$, which leads to an extra factor $2^{2+2\lambda_{q,g}}$.

In Fig. 3 we show a comparison of the exact ratio, derived from Eq. (24), to the approximation in Eq. (26) for the effective distributions in Eq. (25). The change of the integration limit has no remarkable effect up to $\xi \sim 0.1$. The accuracy of the gluon ratio is slightly worse compared to the quark ratio, because of an additional factor of z^2 in the integrand in Eq. (24). Since the quotients of the gamma functions in Eq. (26) have a weak $\lambda_{q,g}$ dependency and Fig. 3 shows a good stability under a change of $\lambda_{q,g}$, we can conclude that Eq. (26) is an excellent approximation of the imaginary part of the amplitudes for the values of ξ, where the effective forward parton distributions can be reliably identified with the conventional forward quark and gluon densities.

The calculation of the real part is straightforward but tedious [we have used Eq. (13) rather than Eq. (17)]. The principal value integration can be performed exactly and the final result consists of integrals without any strong singularities:

$$\mathrm{Re}\,\mathcal{A}_q(\xi) = \int_0^1 \frac{2}{z^2}\left(z + \frac{1}{\sqrt{1+z}} - \frac{1}{\sqrt{1-z}}\right) q_\xi^s\left(\frac{\xi z}{2}\right) dz + \int_{\xi/2x_b}^1 2\left(\frac{1}{z} + \sqrt{\frac{z}{1+z}}\right) q_\xi^s\left(\frac{\xi}{2z}\right) dz, \tag{27a}$$

$$\mathrm{Re}\,\mathcal{A}_g(\xi) = \int_0^1 \frac{4}{z^3}(z^2 - 8 + 4\sqrt{1+z} + 4\sqrt{1-z}) g_\xi\left(\frac{\xi z}{2}\right) dz + \int_{\xi/2x_b}^1 4\left(\frac{1}{z} - 8z + 4\sqrt{z(1+z)}\right) g_\xi\left(\frac{\xi}{2z}\right) dz. \tag{27b}$$

We note that the expressions in brackets in the first integrals in Eq. (27) are always negative, therefore, the two integrals partly cancel each other. Again, we insert the small-x behavior of Eq. (25) into these integrals and set the lower limits of the second integrals to zero, so that everything can be evaluated and yields the following ratios between the real and imaginary parts:

$$R_{q,g}^{\mathrm{Re}} := \frac{\mathrm{Re}\,\mathcal{A}_{q,g}(\xi)}{\mathrm{Im}\,\mathcal{A}_{q,g}(\xi)} \simeq \tan\frac{\pi\lambda_{q,g}}{2}. \tag{28}$$

This is identical to the result, achieved by dispersion relations, in Refs. [21,14]. From Fig. 4 we see that the quality of the approximations of the real parts is significantly more sensitive—note the logarithmic scale—to the change of the integration limit and to a variation of $\lambda_{q,g}$. Additionally, the right-hand side of Eq. (28) depends strongly on $\lambda_{q,g}$, and the first integrals in Eq. (27) have dominant contributions from two regions: around $x \sim 0$ and $x \sim \xi/2$, i.e., we must require that $\lambda_{q,g}$ is essentially constant for small x. Therefore, only for very small ξ, when Eq. (25) is a valid approximation for a large range of x for the usual forward quark and gluon distribution, the latter can be used to reliably predict the real part of the amplitude. Nevertheless, for small values of $\lambda_{q,g}$, where the real part is strongly suppressed, the absolute values of the amplitudes are determined to a good precision for small ξ [14].

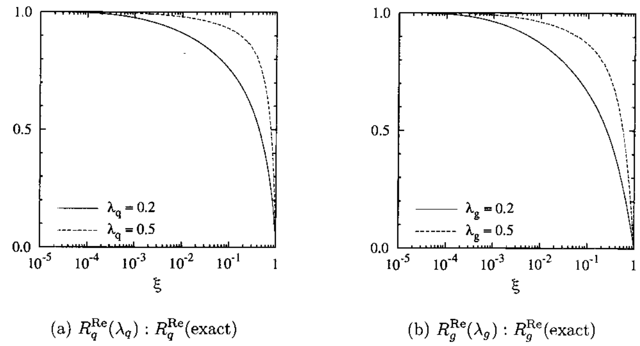

FIG. 4. Comparison of the exact real part to the approximative ratio $R_{q,g}^{\mathrm{Re}}(\lambda_{q,g})$ for $q_\xi^s(x), g_\xi(x) \sim x^{-\lambda_{q,g}-1}$.

VI. MOMENT-DIAGONAL MODELS

In this section, we try to build a model for the effective forward parton distributions. The situation does not seem to be very promising, because Shuvaev's inverse transformation cannot generally be used and we face a difficult support area for the EFPDs in Fig. 2.

References [14,22,18] were dealing with a model for off-forward parton distributions, the Gegenbauer moments of which are independent of ξ. Because the corresponding EFPDs are also independent of ξ, these models manifestly violate the support area in Fig. 2, as was also recognized in Ref. [18], though it should be a good approximation for small values of ξ. This model would have had the great advantage that it would have been stable against a change of the input scale, since the evolution of the Gegenbauer moments is identical to that of the Mellin moments.

Nevertheless, this idea can be used to find valid models for the effective forward parton distributions. Because a common n-, ξ-, or t-dependent factor does not have any influence on the evolution equations, we can generalize the model with ξ-independent Gegenbauer moments to a class of moment-diagonal models with a common proportionality factor:

$$G_n^{q,g}(\xi,t,\mu) := \mathrm{const}(n,\xi,t) \times M_n^{q,g}(\mu). \tag{29}$$

The t dependence is usually factorized, const$(n,\xi)\times F_1(t)$, where $F_1(t)$ is the Dirac form factor [14,22]. These models allow, provided that the Mellin inverse can be performed and leads to valid supports, a direct and simple calculation of off-forward parton distributions and amplitudes, with help of the formulas (17), (24), and (27), for arbitrary x, ξ, μ, and for the region of t, where the $\mathcal{O}(\Delta)$ contributions can be neglected. The simplest model for proton OFPDs of this class, which fulfills all known theoretical constraints for $t=0$ (e.g., see Ref. [5]) and gives a good approximation of the t-dependence, is

$$G_n^{q,g}(\xi,t,\mu) := 2\left(\frac{\xi}{2}\right)^{n+1} T_{n+1}(\xi^{-1}) F_1^p(t) M_n^{q,g}(\mu), \tag{30}$$

where $M_n^{q,g}(\mu)$ are the Mellin moments of the usual quark and gluon distributions in the proton, $F_1^p(t)$ is the Dirac form factor of the proton with $F_1^p(0)=1$, and $T_n(x)$ are Chebyshev polynomials of the first kind {Eq. (22.2.4) in Ref. [16]}. It is advantageous to use the Glück–Reya–Vogt 1998 (GRV 98) parton distributions [10], since they are given for x values down to 10^{-9}, which allows us to compute the real part in Eq. (27) for small values of ξ with a high accuracy. With use of Eq. (22.3.25) and (4.4.27) in Ref. [16], a Mellin inversion yields

$$q_\xi(x)=\left\{\theta\left(\frac{1+\sqrt{1-\xi^2}}{2}-|x|\right)q\left(\frac{2x}{1+\sqrt{1-\xi^2}}\right)+\theta\left(\frac{1-\sqrt{1-\xi^2}}{2}-|x|\right)q\left(\frac{2x}{1-\sqrt{1-\xi^2}}\right)\right\}F_1^p(t), \tag{31a}$$

$$g_\xi(x)=\left\{\theta\left(\frac{1+\sqrt{1-\xi^2}}{2}-|x|\right)g\left(\frac{2x}{1+\sqrt{1-\xi^2}}\right)+\theta\left(\frac{1-\sqrt{1-\xi^2}}{2}-|x|\right)g\left(\frac{2x}{1-\sqrt{1-\xi^2}}\right)\right\}F_1^p(t). \tag{31b}$$

We see that the effective forward parton distributions are a simple combination of two rescaled forward parton densities with the correct support area and an appropriate common t-dependent factor. The first one gives the conventional forward quark and gluons for vanishing ξ and t. The contribution of the second summand is restricted to $|x|<\xi/2$, i.e., it influences only the meson-wave-function-like region $|x|<\xi$ of OFPDs, and is negligible for small values of ξ. Therefore, most of the numerical results of Refs. [14,22,18] can be accurately transferred to the model in Eq. (31).

It is an interesting fact that the argument of the first forward parton density in Eq. (31) is very similar to the Georgi–Politzer ξ-scaling variable [23], $\xi=2x_B/(1+\sqrt{1+4x_B^2M_N^2/Q^2})$, that originally described target mass effects in deep inelastic scattering. Hence, one can argue that the arguments of the parton densities in Eq. (31) reflect skewedness effects. But such arguments can as well be relics of the Gegenbauer polynomials that appear in the derivation [24,25] of the ξ-scaling variable.

VII. SUMMARY AND CONCLUSIONS

In this paper, we presented with Eqs. (17), (24), and (27) simple expressions that relate effective forward parton distributions, which evolve like conventional forward partons, to off-forward parton distributions and nonforward amplitudes. We emphasized that the off-forward parton distributions and nonforward amplitudes can be directly determined from the conventional forward parton distributions and nucleon form factors at arbitrary scale μ for moment-diagonal models. Exemplary, we stated a simple self-consistent model for the EFPDs of the proton in terms of the GRV 98 parton distributions [10] and the Dirac form factor of the proton, which allows us to predict off-forward parton distributions and nonforward amplitudes for arbitrary x, ξ, μ, and (not to large) t. These predictions should not differ too much from results of other models at least at small ξ, as the results in Refs. [14,22] show.

Nevertheless, it would be illuminating if further self-consistent moment-diagonal models exist, especially models that have a qualitatively different behavior in the meson-wave-function-like region, such as the off-forward parton distributions of chiral soliton model calculations [26], since the real part of nonforward amplitudes is dominated by this region and gets important for large ξ.

Because of the complicated support area of EFPDs, an investigation of the double distributions of moment-diagonal models might be helpful.

ACKNOWLEDGMENTS

We thank M. Glück and E. Reya for proposing this investigation and for instructive remarks during the initial stage of this work.

[1] F.-M. Dittes, D. Müller, D. Robaschik, B. Geyer, and J. Hořejši, Phys. Lett. B **209**, 325 (1988).
[2] D. Müller, D. Robaschik, B. Geyer, F.-M. Dittes, and J. Hořejši, Fortschr. Phys. **42**, 101 (1994).
[3] X. Ji, Phys. Rev. Lett. **78**, 610 (1997).
[4] X. Ji, Phys. Rev. D **55**, 7114 (1997).
[5] X. Ji, J. Phys. G **24**, 1181 (1998).
[6] J. C. Collins, L. Frankfurt, and M. Strikman, Phys. Rev. D **56**, 2982 (1997).
[7] A. V. Radyushkin, Phys. Lett. B **385**, 333 (1996).
[8] A. V. Radyushkin, Phys. Rev. D **56**, 5524 (1997).
[9] A. V. Radyushkin, Phys. Lett. B **380**, 417 (1996).
[10] M. Glück, E. Reya, and A. Vogt, Eur. Phys. J. C **5**, 461 (1998).
[11] H. L. Lai, J. Huston, S. Kuhlmann, J. Morfin, F. Olness, J. F. Owens, J. Pumplin, and W. K. Tung, Eur. Phys. J. C **12**, 375 (2000).
[12] A. D. Martin, R. G. Roberts, W. J. Stirling, and R. S. Thorne, Eur. Phys. J. C **14**, 133 (2000).
[13] A. Shuvaev, Phys. Rev. D **60**, 116005 (1999).
[14] A. G. Shuvaev, K. J. Golec-Biernat, A. D. Martin, and M. G. Ryskin, Phys. Rev. D **60**, 014015 (1999).
[15] M. V. Polyakov and C. Weiss, Phys. Rev. D **60**, 114017 (1999).
[16] *Handbook of Mathematical Functions*, 9th ed., edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[17] A. V. Radyushkin, Phys. Lett. B **449**, 81 (1999).
[18] I. V. Musatov and A. V. Radyushkin, Phys. Rev. D **61**, 074027 (2000).
[19] B. C. Carlson, Math. Comput. **53**, 327 (1989).
[20] B. C. Carlson, Numer. Math. **33**, 1 (1979).
[21] L. L. Frankfurt, A. Freund, and M. Strikman, Phys. Rev. D **58**, 114001 (1998); **59**, 119901(E) (1999).

[22] K. J. Golec-Biernat, A. D. Martin, and M. G. Ryskin, Phys. Lett. B **456**, 232 (1999).
[23] H. Georgi and H. D. Politzer, Phys. Rev. D **14**, 1829 (1976).
[24] O. Nachtmann, Nucl. Phys. **B63**, 237 (1973).
[25] R. Barbieri, J. Ellis, M. K. Gaillard, and G. G. Ross, Nucl. Phys. **B117**, 50 (1976).
[26] V. Yu. Petrov, P. V. Pobylitsa, M. V. Polyakov, I. Börnig, K. Goeke, and C. Weiss, Phys. Rev. D **57**, 4325 (1998).